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Relative dispersion and quantum thermal equilibrium in de Broglie–Bohm mechanics

Andrew F Bennett

College of Oceanic and Atmospheric Sciences, Oregon State University, Corvallis, OR 97331-5503, USA

E-mail: bennett@coas.oregonstate.edu

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Abstract

Numerical computations (Valentini and Westman 2005 *Proc. R. Soc.* A **461** 253–72) demonstrate that an initially arbitrary particle density, stirred by the field of de Broglie velocity associated with the Schrödinger wavefunction for a sum of energy eigenstates, relaxes to the quantum thermal equilibrium that is the Born probability density, provided the particle density is coarse grained. The results are explained here by a Lagrangian or trajectory analysis, in terms of the relative dispersion of passive particles in a turbulent fluid. The analysis assumes that the turbulence statistics are stationary and isotropic, although these assumptions may be weakened. The relaxation to equilibrium is not reversible, owing to the coarse graining of the particle density and to the statistical inevitability of particle separation. There is no effective stirring toward equilibrium in very simple quantum systems such as a Gaussian wave packet or an energy eigenstate. However, it is argued that relaxation takes place during the emission of the packet or the establishment of the eigenstate, owing to stirring by the transients in the wavefunction for the entire system. The Lagrangian analysis is readily extended to nonrelativistic many-particle systems and to relativistic single-particle systems.

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1. Introduction

1.1. Preamble

The purpose of this investigation is the elimination of a fundamental assumption in de Broglie–Bohm mechanics. The assumption is instead derived from the other fundamental assumptions, using analytical methods borrowed from classical fluid dynamics. The detailed derivation is presented for the simplest case of a single-particle, nonrelativistic quantum system in a state that is a sum of energy eigenstates. The derivation is outlined for more complicated systems. The assumption at issue asserts that the spatial distribution of the position of a particle,

the position being uncertain solely due to a lack of empirical knowledge, is the same as the Born probability distribution of orthodox quantum mechanics. The assumption is eliminated by proving that the coarse-grained particle density relaxes in time to the Born density, that is, the latter is the 'quantum thermal equilibrium' for the former. There are other arguments for density relaxation, which either introduce a level of motion below that associated with the quantum system [1] or which ignore coarse graining but instead introduce a concept of 'typicality' along with the notion of the wavefunction of the universe [2, 3]. The latter argument assumes that the initial configuration of the universe is random and distributed according to the initial Born density of the universe. The proof of relaxation given here has no such requirements. Quantum mechanics is well known, yet it is necessary to review some of its bare essentials in order to describe de Broglie–Bohm mechanics, and hence the plan of this paper.

1.2. de Broglie–Bohm mechanics

de Broglie–Bohm mechanics is a theory for the motion of subatomic particles [4–6]. Like orthodox quantum mechanics, to which it is a putative alternative, de Broglie–Bohm mechanics or 'dBB' expresses quantitative information about a particle in terms of a 'wavefunction' ψ, which is a field over position $\mathbf{x}$ in three-dimensional space $\mathbb{R}^3$ and which is a continuous function of time t. The wavefunction ψ is the solution of a 'wave equation' subject to initial and boundary values. In the simplest case, the nonrelativistic wave equation for a single particle of mass m and spin-0 is the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi, \tag{1}$$

where $\psi = \psi[\mathbf{x}, t]$ takes complex values in $\mathbb{C}$, $V = V[\mathbf{x}]$ is an externally imposed and real-valued potential field, while $\hbar$ is the reduced Planck's constant (see, e.g., [7]). The Laplacian operator ∇^2 is $\nabla\cdot\nabla$ where the gradient operator ∇ is, in component form, $\partial/\partial x_i$, the components of $\mathbf{x}$ being x_i for $i = 1, 2, 3$. According to orthodox quantum mechanics ('QM'), the particle has no definite quantitative attributes in general. Rather, measurements of the particle yield at random the eigenvalues of a Hermitian operator acting upon ψ, there is a particular operator for each kind of measurement. The random values take their probability distribution function from the Born density $|\psi|^2$. For example, the particle has no definite position. The QM expectation value of a position measurement is

$$\langle\mathbf{x}\rangle[t] = \int_D \psi^*[\mathbf{x}, t]\mathbf{x}\psi[\mathbf{x}, t]\,\mathrm{d}^3\mathbf{x}, \tag{2}$$

where D is the spatial domain of ψ. The wavefunction is normalized so that $\int_D |\psi|^2\mathrm{d}^3\mathbf{x} = 1$. The particle has no definite momentum either, and the QM expectation value of a momentum measurement is, by invoking the de Broglie relationship $\mathbf{p} = -\mathrm{i}\hbar\nabla$ between momentum and wave number,

$$\langle\mathbf{p}\rangle[t] = -\mathrm{i}\hbar\int_D \psi^*[\mathbf{x}, t]\nabla\psi[\mathbf{x}, t]\,\mathrm{d}^3\mathbf{x}. \tag{3}$$

That $\langle\mathbf{p}\rangle$ has real values is a consequence of Green's theorem and the boundary values for ψ. The justification for adopting $|\psi|^2$ as a probability density is that solutions of (1) obey the Born identity

$$\frac{\partial|\psi|^2}{\partial t} + \nabla\cdot\mathbf{j} = 0, \tag{4}$$

where the 'probability current' $\mathbf{j} = \mathbf{j}[\mathbf{x}, t]$ is defined by

$$\mathbf{j} = (\hbar/m)\mathrm{Im}(\psi^*\nabla\psi). \tag{5}$$

It is evident from (4) that $\int_D |\psi|^2 \mathrm{d}^3\mathbf{x}$ is constant in time, provided $\psi^* \nabla \psi$ is real on the boundary. This condition is satisfied by a variety of natural choices of boundary conditions for (1). The normalization is also constant if ψ is periodic in space [8].

According to dBB, on the other hand, the particle has some definite position $\mathbf{x}$ at any time t. It also has a definite momentum $m\mathbf{v}$, where the de Broglie velocity $\mathbf{v}$ is defined by the ratio of the probability current and probability density:

$$\mathbf{v}[\mathbf{x}, t] \equiv \mathbf{j}[\mathbf{x}, t]/|\psi[\mathbf{x}, t]|^2. \tag{6}$$

Expressing the wavefunction in the polar form $\psi = |\psi| \exp(\mathrm{i}S/\hbar)$, where the unit of phase S is the reduced quantum of action $\hbar$, it follows readily that the momentum of the particle is the gradient of the phase:

$$m\mathbf{v} = \nabla S. \tag{7}$$

Consider, for example, the spherical wavefunction for a free α-particle [9]:

$$\psi[\mathbf{x}, t] = |\mathbf{x}|^{-1} \exp(\mathrm{i}k|\mathbf{x}| - \mathrm{i}\omega t), \tag{8}$$

with k being a constant and $\omega = \hbar k^2/2m$. The de Broglie velocity is

$$\mathbf{v}[\mathbf{x}, \mathbf{t}] = (\hbar k/m)\frac{\mathbf{x}}{|\mathbf{x}|}, \tag{9}$$

that is, the α-particle moves steadily along some ray through the origin.

The current $\mathbf{j}$ and hence the de Broglie velocity $\mathbf{v}$ may be defined for an arbitrary Hermitian Hamiltonian [10]. Quantum spin adds a rotational part to the de Broglie velocity [11–13].

If a particle has position $\mathbf{a}$ at time s, then its position $\mathbf{x}$ at another time t is given in general by the path function $\mathbf{x} = \mathbf{P}(\mathbf{a}, s; t)$, where $\mathbf{P}$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}(\mathbf{a}, s; t) = \mathbf{v}[\mathbf{P}(\mathbf{a}, s; t), t] \tag{10}$$

subject to the general initial condition

$$\mathbf{P}(\mathbf{a}, s; s) = \mathbf{a}. \tag{11}$$

The α-particle with de Broglie velocity (9) has the path $\mathbf{x} = \mathbf{a}t/s$, provided $|\mathbf{a}|/s = \hbar k/m$.

It being a practical impossibility to measure the position of a particle perfectly, there is imprecision in $\mathbf{a}$ and subsequently in $\mathbf{x}$. This imprecision is expressed as a particle density $\rho = \rho[\mathbf{x}, t]$, or relative frequency of finding the particle near $\mathbf{x}$ at time t. On the assumption that particles are neither created nor destroyed, the density evolves according to the conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0. \tag{12}$$

It is convenient to normalize the particle density so that $\int_D \rho \, \mathrm{d}^3\mathbf{x} = 1$. Thus, the dBB expectation value of position at time t is

$$\langle \mathbf{x} \rangle_\rho[t] = \int_D \mathbf{x}\rho[\mathbf{x}, t] \, \mathrm{d}^3\mathbf{x}, \tag{13}$$

while the dBB momentum expectation is

$$\langle m\mathbf{v} \rangle_\rho[t] = m \int_D \mathbf{v}[\mathbf{x}, t]\rho[\mathbf{x}, t] \, \mathrm{d}^3\mathbf{x}. \tag{14}$$

The de Broglie velocity field $\mathbf{v}$ being defined by (6), the identity (4) may be written in the form

$$\frac{\partial |\psi|^2}{\partial t} + \nabla \cdot (\mathbf{v}|\psi|^2) = 0. \tag{15}$$

The solutions of (12) and (15) are identical everywhere at any time t, if and only if they are identical at one time s. If ρ and $|\psi|^2$ are the same at some time, then the expectation values predicted by dBB and by QM, respectively, for that time are also the same. This is trivially obvious for the position (compare (2) with (13)) and is readily seen to be the case also for momentum (compare (3) with (14)). Consider again the example [9] of the free α-particle with de Broglie velocity (9). It is clear that, even though the Born density $|\psi|^2$ cannot be normalized, the assumed equality of shapes of $|\psi|^2$ and ρ ensures that the most probable initial position of the particle is close to the origin. For a complete discussion of expectation values, and also the dBB account of the Heisenberg uncertainty principle, see [11, 14].

The requirement that ρ and $|\psi|^2$ be initially and therefore subsequently identical is perhaps more than anything else responsible for the lack of widespread preference for dBB over QM, even though dBB does not lead to the 'measurement problem' of QM, and even though the mystery found in QM over 'delayed choice' experiments does not arise in dBB [1, 5, 6, 11][1]. The vexatious assumption of initial agreement between ρ and $|\psi|^2$ may be discarded, if it can be proven that $\rho \sim |\psi|^2$ after a few oscillations of the quantum system, that is, if $|\psi|^2$ is the quantum thermal equilibrium (QTE) for the system. The demonstration by Valentini and Westman [8] of the relaxation of the 'coarse-grained' particle density to QTE is a major development for dBB theory. Valentini [15] has proposed that photons emitted in the early universe may not yet be in QTE. Again, relaxation to QTE can only be attributed to the stirring of the particle density by the field of de Broglie velocity. The dBB continuum field theory in which this effect is to be established strikingly resembles the classical theory of a compressible fluid [16, 17].

1.3. Fluid dynamics

There are advantages to the 'Lagrangian' picture of fluid kinematics over the 'Eulerian' picture, and these advantages are used here to establish relaxation. It must first be understood that it is meaningless to speak of the density defined at a single point. Rather, the density is the total number of particles in a small but finite volume containing the point. Thus, relaxation must owe to the history of a small volume element, rather than that of a single point. In fluid-mechanical terminology, relaxation owes to *relative* dispersion and not to *absolute* dispersion[2]. The general principles of relative dispersion are well understood, and it suffices here to consider only the stretching of an infinitesimal line element. The analysis is foreshadowed by Bohm and Hiley [1]. The phase complexity of many quantum states and the numerical evidence that dBB particle paths become chaotic support the use of formal statistical arguments concerning relative dispersion in dBB. The statistics are taken from the small cell that defines the locally averaged or coarse-grained density, and also from the entire domain.

The dBB velocity field for a sum of energy eigenstates is regarded here as a random process that is statistically stationary in time and statistically isotropic in space. The condition of statistical isotropy implies statistical homogeneity in space. These assumptions are reasonable when the quantum state is an incoherent or random sum of many energy eigenstates, and the domain is of simple shape. It is assumed that the time series of Lagrangian velocities and

[1] The lectures of Feynman *et al* [7] on quantum mechanics open with what is often cited as the definitive statement of orthodox quantum mechanics, in the context of the Young double-slit experiment. The epilogue to the lectures, on the other hand, is a painstaking development of de Broglie–Bohm mechanics as a description of electron flow in a superconductor.

[2] The latter is of interest in the study of turbulent mixing, in which there is understood to be an ensemble of velocity fields. Hence a particle released at a single point will have an ensemble of subsequent positions. The flow ensemble may be real or may be the ideal representation of flows in a sequence of disjoint time intervals, throughout all of which the sample statistics are stationary. There is only one velocity field here, the single field of de Broglie velocity.

those of their spatial gradients, or in other words the histories of velocities and gradients following the motion, decorrelate in time. It is also assumed that the lagged autocorrelation functions are integrable over the lags. The last-mentioned assumption ensures the realizability of the random Lagrangian processes. The assumption of stationarity simplifies the evaluation of the variance of a time integral following the motion and may be weakened. Provided the classical Lindeberg condition holds, the integral is asymptotically normal for large time magnitude and the separation of particle pairs is log-normal. The assumption of isotropy may be weakened to homogeneity, so long as the relative-diffusivity tensor remains positive-definite. A further inference of 'whiteness' is made for the spatial field of initial 'defect' or initial relative departure from QTE, in the following sense. It is shown below that defect necessarily has a vanishing spatial average at all times and is conserved following the motion. It is also shown below that particles separation variance increases as time increases in either sense. Thus particles in the same coarse-graining cell at some late time can be expected, at the initial time, to be randomly and widely scattered into the several subdomains in which the defect takes one sign or the other with equal measure. The initial defect values sampled by the particles are accordingly uncorrelated from particle to particle. It immediately follows that the coarse-grained defect at the later time not only has identically vanishing expectation but also has asymptotically small variance. That is, QTE obtains asymptotically.

In the terminology of quantum mechanics, the state is nonstationary since its wavefunction is a linear combination of time-dependent and mutually interfering energy eigenstates. The terminology of time series is to be understood subsequently. The quantum statistics of the state are pure since it is represented by a single wavefunction. If the quantum statistics were mixed, meaning that the quantum system is represented by several wavefunctions each of which has some relative frequency, then the analogy with turbulence would be even stronger. For a further discussion of dBB quantum statistics, see [1] and [11]. Returning to the pure state considered here, the assumptions of isotropic and stationary turbulence in the dBB velocity field make the finding of irreversible relaxation to QTE all the more striking.

The plan of the rest of this paper is as follows. The formalities of particle kinematics are outlined in section 2. The rapid growth of fine structure in particle paths and the attendant need for local spatial averaging or coarse graining are established in section 3. The main result, namely the relaxation of coarse-grained particle densities to QTE, is also derived in the outline in section 3. The sampling statistics that quantify relaxation are developed in detail in section 4 and are followed in section 5 by a number of remarks on the various assumptions made in this analysis. The irreversibility of relaxation is described in section 6, together with a brief discussion of the paradox presented by the Poincaré recurrence theorem. Several simple counterexamples, in which relaxation does not occur, are described in section 7 but they are shown to be incomplete in the sense that they do not take into account the processes by which the simple quantum systems are established. It is however explained, in terms of spectral coherence, why relaxation may fail to occur in these counterexamples. There are straightforward extensions of the proof of relaxation for multi-particle systems and for relativistic systems, and the extensions are discussed in outline in section 8.

2. Particle kinematics

2.1. Differential equations

Consider again a particle located at the position $\mathbf{a}$ at time s. Denote the position $\mathbf{x}$ of the particle at another time t by the path function $\mathbf{x} = \mathbf{P}(\mathbf{a}, s; t) \in \mathbb{R}^3$. In particular, $\mathbf{P}(\mathbf{a}, s; s) = \mathbf{a}$. The parameters $\mathbf{a}$ and s serve to 'label' the path passing through the point $\mathbf{a}$ at time s. The Eulerian

field $\mathbf{v} = \mathbf{v}[\mathbf{x}, t]$ of the de Broglie velocity (6) determines particle paths $\mathbf{x} = \mathbf{P}(\mathbf{a}, s; t)$ via the Lagrangian prescription (10) subject to the general initial condition (11). In the interest of clarity, Eulerian arguments are denoted as $[\mathbf{x}, t]$ and Lagrangian arguments as $(\mathbf{a}, s; t)$. Note that both $t > s$ and $t < s$ are allowed. For each well-behaved field of velocity $\mathbf{v}$, the path $\mathbf{P}$ is indeed a function of its arguments. In particular, the particle position $\mathbf{x}$ at time t is unique [18], given that it is at $\mathbf{a}$ at time s. Interchanging symbols, $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$ and $\mathbf{P}(\mathbf{x}, t; t) = \mathbf{x}$. The path $\mathbf{P}$ is, in this sense, its own inverse: $\mathbf{P}(\mathbf{P}(\mathbf{a}, s; t), t; s) = \mathbf{a}$. To exchange the Lagrangian and Eulerian 'pictures', note that for any function F, it is the case that $F(\mathbf{a}, s; t) = F[\mathbf{P}(\mathbf{a}, s; t), t]$ and $F[\mathbf{x}, t] = F(\mathbf{P}(\mathbf{x}, t; s), s; t)$. It is readily established (see, e.g., [19]) that the second of these identities is independent of the labeling time s. It would be more consistent to denote the time derivative in (10) by a partial derivative, as in [19], but convention is followed here. The time derivative in (10) is said to 'follow the motion' of the particle. The Jacobian of the transformation $\mathbf{a} \to \mathbf{P}(\mathbf{a}, s; t)$ is denoted by $J = J(\mathbf{a}, s; t)$, where

$$J = \frac{\partial(P_1, P_2, P_3)}{\partial(a_1, a_2, a_3)}. \tag{16}$$

If $\mathbf{x} = \mathbf{P}(\mathbf{a}, s; t)$, then $J(\mathbf{a}, s; t)J(\mathbf{x}, t; s) = J(\mathbf{a}, s; s) = 1$. It is a kinematic identity (see, e.g., [20]) that

$$\frac{\mathrm{d}J}{\mathrm{d}t}(\mathbf{a}, s; t) = J(\mathbf{a}, s; t)\nabla \cdot \mathbf{v}[\mathbf{P}(\mathbf{a}, s; t), t], \tag{17}$$

where $\nabla \cdot \mathbf{v}[\mathbf{x}, t] = (\partial v_i/\partial x_i)[\mathbf{x}, t]$ is the flow divergence.

The Lagrangian and Eulerian expressions, for the rate of change of a field F following the motion, are related by

$$\frac{\mathrm{d}F}{\mathrm{d}t}(\mathbf{a}, s; t) = \frac{\partial F}{\partial t}[\mathbf{x}, t] + \mathbf{v}[\mathbf{x}, t] \cdot \nabla F[\mathbf{x}, t], \tag{18}$$

where $\mathbf{x} = \mathbf{P}(\mathbf{a}, s; t)$.

According to the dBB theory, the particle density $\rho = \rho[\mathbf{x}, t]$ obeys the Eulerian conservation law (12). The law may be rearranged as

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v}, \tag{19}$$

which by virtue of (17)–(19) has [20] the Lagrangian form

$$\frac{d(\rho J)}{\mathrm{d}t}(\mathbf{a}, s; t) = 0, \tag{20}$$

where $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$. In the same way, (15) has the Lagrangian form

$$\frac{d(|\psi|^2 J)}{\mathrm{d}t}(\mathbf{a}, s; t) = 0. \tag{21}$$

2.2. *First integrals*

The solution of (21) may be expressed as

$$|\psi|^2[\mathbf{x}, t]J(\mathbf{a}, s; t) = |\psi|^2[\mathbf{a}, s], \tag{22}$$

where $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$. Combining (22) with the corresponding first integral of (20) yields

$$f[\mathbf{x}, t] \equiv \frac{\rho[\mathbf{x}, t]}{|\psi|^2[\mathbf{x}, t]} = \frac{\rho[\mathbf{a}, s]}{|\psi|^2[\mathbf{a}, s]} = f[\mathbf{a}, s], \tag{23}$$

that is, the density ratio $f \equiv \rho/|\psi|^2$ is conserved following the motion. It will be convenient to define the 'defect' $g[\mathbf{x}, t] \equiv f[\mathbf{x}, t] - 1$; thus $g[\mathbf{x}, t] = g[\mathbf{a}, s]$ where, again, $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$.

2.3. Conserved expectations

The conservation of $|\psi|^2 J$ on particle paths implies that the QM measure is also conserved, in the sense that

$$|\psi|^2[\mathbf{x}, t]\,\mathrm{d}^3\mathbf{x} = |\psi|^2[\mathbf{a}, s]\,\mathrm{d}^3\mathbf{a}, \tag{24}$$

when $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$. Specializing to $s = 0$, it is immediately the case that for any field which is conserved on particle paths, such as the density ratio f, and for any function $\mathcal{F}(f)$,

$$\mathcal{F}(f[\mathbf{x}, t])|\psi|^2[\mathbf{x}, t]\,\mathrm{d}^3\mathbf{x} = \mathcal{F}(f_0[\mathbf{a}])|\psi_0|^2[\mathbf{a}]\,\mathrm{d}^3\mathbf{a}, \tag{25}$$

where $f_0[\mathbf{a}] = f[\mathbf{a}, 0]$ and $\psi_0[\mathbf{a}] = \psi[\mathbf{a}, 0]$. Hence, the QM expectation of $\mathcal{F}(f)$ is conserved:

$$\langle \mathcal{F}(f)\rangle = \langle \mathcal{F}(f_0)\rangle. \tag{26}$$

For example, taking $\mathcal{F}(f) \equiv f - 1 = g$ and $\mathcal{F}(f) \equiv (f-1)^2 = g^2$, respectively,

$$\langle g\rangle = \langle g_0\rangle = \int_D (\rho_0[\mathbf{a}] - |\psi_0|^2[\mathbf{a}])\,\mathrm{d}^3\mathbf{a} = 0, \tag{27}$$

and

$$\langle g^2\rangle = \langle g_0^2\rangle = \int_D \frac{(\rho_0[\mathbf{a}] - |\psi_0|^2[\mathbf{a}])^2}{|\psi_0|^2[\mathbf{a}]}\,\mathrm{d}^3\mathbf{a}, \tag{28}$$

where $g_0[\mathbf{a}] = g[\mathbf{a}, 0]$, $\rho_0[\mathbf{a}] = \rho[\mathbf{a}, 0]$. There is an integrable singularity in (28) if ψ_0 has a first-order node or simple zero.

The vanishing QM expectation for g_0, as expressed in (27), implies that the initial defect $g_0[\mathbf{a}]$ takes both positive and negative values in the domain D. The variance of these values is finite, as expressed in (28). It will be shown in section 4 that at any time t, a coarse graining or local averaging of the field of defect $g[\mathbf{x}, \mathbf{t}]$ is a sample mean of a large number of unbiased samples of the initial defect field $g_0[\mathbf{a}]$. The lack of bias owes to the relative dispersion of the sampling points in the coarse-graining cell, backward in time, and to the particle paths being independent of the initial defect. It will also be shown that the sample mean has vanishing QM expectation, and small QM variance of $\mathcal{O}(N^{-3})$ for N^3 samples. The vanishing expectation and small variance of g, after coarse graining, are equivalent to relaxation of the dBB density ρ to the Born density $|\psi|^2$, since the latter density is not significantly affected by the coarse graining.

3. Coarse graining and relaxation

3.1. Fine structure in ρ

The numerical computations of Valentini and Westman [8] yield evolved fields of particle density ρ for $t > s$, having a very fine structure not present at $t = s$. Further such computations are reported in [21]. In all those computations $s = 0$, but it is necessary here to retain briefly a variable s. The 'fine grain' found in the computations is a consequence of the relative dispersion of initially neighboring particles, as will now be shown. Rearranging (23),

$$\rho[\mathbf{x}, t] = |\psi|^2[\mathbf{x}, t] f[\mathbf{a}, s], \tag{29}$$

where $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$. The wave equation (1) is linear, so in the presence of a constant potential V, the wavefunction ψ is exactly as smooth spatially at time $t > s$ as it is at time $t = s$, while ψ remains about that smooth when in the presence of a smooth potential. The initial ratio $f[\mathbf{a}, s]$ is smooth in $\mathbf{a}$ if the initial particle density $\rho[\mathbf{a}, s]$ and initial Born density $|\psi|^2[\mathbf{a}, s]$ are smooth, except at nodes of ψ. However, for large values of elapsed time t, the initial

value $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$ of the path function may be very sensitive to the labeling position $\mathbf{x}$. Differentiating (29) with respect to x_i yields

$$\frac{\partial \rho}{\partial x_i}[\mathbf{x}, t] = \frac{\partial |\psi|^2}{\partial x_i}[\mathbf{x}, t] f[\mathbf{a}, s] + |\psi|^2[\mathbf{x}, t] \frac{\partial f}{\partial a_p}[\mathbf{a}, s] \frac{\partial P_p}{\partial x_i}(\mathbf{x}, t, s), \tag{30}$$

where again $\mathbf{a} = \mathbf{P}(\mathbf{x}, t; s)$. Changing the symbols $(\mathbf{a}, s; t)$ in (10) to $(\mathbf{x}, t; s)$, and differentiating with respect to the label x_i yields

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial P_p}{\partial x_i}(\mathbf{x}, t; s) = \frac{\partial v_p}{\partial x_q}[\mathbf{P}(\mathbf{x}, t; s), s] \frac{\partial P_q}{\partial x_i}(\mathbf{x}, t; s), \tag{31}$$

subject to $(\partial P_p/\partial x_i)(\mathbf{x}, t; t) = \delta_{pi}$. In local coordinates rotating at the rate $\omega_p/2$, where $\omega_p \equiv \epsilon_{pqr} \partial v_r/\partial x_q$ is a component of the vorticity in the de Broglie velocity field, (31) becomes (without introducing a new symbol for the rotating coordinates, and suppressing the arguments for clarity)

$$\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial P_p}{\partial x_i} = e_{pq} \frac{\partial P_q}{\partial x_i}, \tag{32}$$

where e_{pq} is the symmetric part of the rate-of-strain tensor $\partial v_p/\partial x_q$. Note that the de Broglie velocity field is rotational in the presence of quantum spin, in which case $\partial v_p/\partial x_q$ is not symmetric. The eigenvalues of e_{pq} are all real, and their sum is the divergence $\nabla \cdot \mathbf{v}$ of the de Broglie velocity field. The Eulerian integral of the divergence, over the rigidly or periodically bounded domain D, vanishes at all times:

$$\int_D \nabla \cdot \mathbf{v}[\mathbf{x}, t] \, \mathrm{d}^3\mathbf{x} = 0; \tag{33}$$

thus, regions having positive values for $\nabla \cdot \mathbf{v}$ (divergences) and regions having negative values (convergences) occur simultaneously. We are concerned with integrating (32) from $s = t > 0$ back to $s = 0$. We are therefore especially interested in convergences, which yield larger gradients for conserved quantities since e_{pq} has at least one negative eigenvalue at a convergence (points that are close at the present time are further apart in the past). The rapid temporal and spatial variability of the phase S in regions of interference yield equally rapid sign changes for its Laplacian $\nabla^2 S$ and hence for $\nabla \cdot \mathbf{v}$. Thus, particles most likely travel through many convergences and divergence during long 'flights'.

3.2. Infinitesimal line stretching

It will be shown below that moments of path gradients over the domain D have the same asymptotic behavior for forward integration as for backward, so the coordinates $(\mathbf{x}, t; s)$ for the backward problem in (31) and in (32) may as well be replaced with the coordinates $(\mathbf{a}, s; t)$ for the more familiar forward problem. It is appropriate to specialize henceforth to $s = 0$. After diagonalization, (32) becomes (suspending the summation convention)

$$\frac{\mathrm{d}}{\mathrm{d}t} \Gamma_p(t) = \lambda_p(t) \Gamma_p(t), \tag{34}$$

with $\Gamma_p(0) = 1$, where λ_p is the time series of one of the three eigenvalues of e_{pq} on a path and Γ_p is the amplitude of the associated eigenvector. The skew part of the rate-of-strain tensor $\partial v_p/\partial x_q$ being suppressed, the model (34) captures the change in the magnitude of the partial derivatives of the path position but not their rotation. The labels $\mathbf{a}$ and 0 are suppressed for clarity. It will be also be helpful to suppress the subscript p momentarily.

The degrees of freedom in λ may be counted as follows [22]. The moments of λ over D are assumed to be stationary with respect to time, and time reflection invariant. Almost no

trajectories detach from the boundary, the latter being either rigid or periodic, and almost no trajectories divide [18], so uniformly weighted Eulerian moments over D are also Lagrangian moments over D weighted with the Jacobian. The first moment of any field F over D (a cube of side L) is denoted as

$$\overline{F[\mathbf{x},t]} \equiv L^{-3}\int_D F[\mathbf{x},t]\,\mathrm{d}^3\mathbf{x} = L^{-3}\int_D F(\mathbf{a},0;t)J(\mathbf{a},0;t)\,\mathrm{d}^3\mathbf{a} = \overline{F(\mathbf{a},0;t)}, \tag{35}$$

where $\mathbf{a} = \mathbf{P}(\mathbf{x},t;0)$.

Denote first and second moments of the eigenvalues of e_{pq} by $\mu \equiv \overline{\lambda(t)}$ and $\sigma_\lambda^2 \equiv \overline{(\lambda(t)-\mu)^2}$. The Lagrangian statistics of velocity and all derived fields having been assumed stationary, all single-time moments such as μ and σ_λ^2 are constant; therefore,

$$\overline{\int_0^t \lambda(r)\,\mathrm{d}r} = \mu t \tag{36}$$

and

$$\overline{\left(\int_0^t \lambda(r)\,\mathrm{d}r\right)^2} = \mu^2t^2 + 2\sigma_\lambda^2\int_0^{|t|}(|t|-r)C_\lambda(r)\,\mathrm{d}r, \tag{37}$$

where $C_\lambda(|t|)$ is the autocorrelation coefficient for $\lambda(t)$. In particular $C_\lambda(0) = 1$. Note that (36) and (37) allow for $t < 0$. It is assumed in (37) that $C_\lambda(|t|) \to 0$ sufficiently rapidly as $|t| \to \infty$, being the essential realizability condition for a stationary random process [23]. Hence the necessarily positive integral on the right-hand side of (37) has the asymptote $T_\lambda|t|$, where $T_\lambda = \int_0^\infty C_\lambda(t)\mathrm{d}t > 0$ is the 'Lagrangian integral time scale' for the decorrelation of λ following the motion. The number of degrees of freedom in λ in the interval between 0 and t is estimated to be $|t|/T_\lambda$. Assuming that this number is sufficiently large, the integral $I = \int_0^t \lambda(r)\mathrm{d}r$ is, as a consequence of the central limit theorem (see, e.g., [24, 25]), asymptotically normally distributed with moments $\overline{I} = \mu t$ and $\overline{(I-\overline{I})^2} \sim 2\sigma_\lambda^2 T_\lambda|t|$. The absolute asymptotic departure from the normal distribution is uniformly bounded by $\sqrt{T_\lambda/t}$. Hence the generic solution $\Gamma = \exp(I)$ for (34) is lognormal, with the first moment

$$\overline{\Gamma(t)} = \exp\left(\mu t + \sigma_\lambda^2 T_\lambda|t|\right). \tag{38}$$

If $\mu t < 0$, the asymptotic behavior of $\overline{\Gamma(t)}$ for large $|t|$ would seem not to be determined by (38) without a knowledge of $\sigma_\lambda^2 T_\lambda/|\mu|$. Indeed, denoting the three eigenvalues of the symmetrized rate-of-strain tensor e_{pq} once more by $\lambda_p = \lambda_p(t)$, their sum is $\lambda_1+\lambda_2+\lambda_3 = \boldsymbol{\nabla}\cdot\mathbf{v}$ which, again, may be positive or negative. However, since the domain is rigid or periodic, the mean divergence $\overline{\boldsymbol{\nabla}\cdot\mathbf{v}}$ vanishes and hence the sum of the mean eigenvalues also vanishes: $\mu_1+\mu_2+\mu_3 = 0$. It is therefore assured that two of the three μ_p have opposite signs. Thus for at least one value of p, $\overline{\Gamma_p(t)}$ grows exponentially as $t \to \infty$ as do all higher moments of this $\Gamma_p(t)$, while there is for at least one other p exponential growth in $\overline{\Gamma_p(t)}$ and all higher moments as $t \to -\infty$. It is concluded that the mean magnitude $\overline{\left(\sum_p \Gamma_p^2\right)^{1/2}}$, for the solution of the model (34) of the evolution of path gradients (31), grows exponentially both backward in time and forward in time. The fine grain that develops in the particle density ρ owes to this infinitesimal line stretching: $\mathrm{d}x_j \to \mathrm{d}P_i$, or exponentially growing separation of neighboring points, as time runs backward or forward. It is therefore evident from (30) that the spatial fine grain in the density ratio $f[\mathbf{x},t]$ is inevitable if the initial ratio $f[\mathbf{a},s]$ is not uniform with respect to $\mathbf{a}$, no matter how smooth the initial nonuniformity. Numerical computations of dBB paths typically develop chaotic behavior [8, 17, 26–28].

The Born probability current law (15) may be expressed in the Lagrangian form (21), but an initially smooth Born density $|\psi|^2$ remains smooth. It is being assumed that the wavefunction ψ at the initial time $t = s$ is a finite sum of the smooth eigenfunctions of the Hamiltonian in (1). Then $\nabla \cdot \mathbf{j} = (\mathrm{i}/\hbar)(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*)$ is a finite sum of products of these eigenfunctions, and by (4) so also is $|\psi|^2$ at time $s + \mathrm{d}t$. The particle density ρ, on the other hand, has the initial flux divergence $\nabla \cdot (\rho \mathbf{v}) = \nabla \cdot (f \mathbf{j})$ which is in general an infinite sum of products of eigenfunctions, as is ρ at time $s + \mathrm{d}t$.

The preceding analysis of infinitesimal line stretching explicitly assumes that the gradient of the phase S and hence the de Broglie velocity $\mathbf{v}$ are highly variable both in space and time, as is the case when ψ is a sum of many energy eigenfucntions. The assumption of time reflection invariance for the autocorrelation functions of the eigenvalues $\lambda_p(t)$ is solely for simplicity in the evaluation of the squared integral in (37) and has no significant influence on the general finding of expected exponential growth for path gradients as time increases or decreases. That finding depends critically upon the vanishing of the divergence $\nabla \cdot \mathbf{v}$, averaged with weight J over all the paths in the domain. In other words, the sample set of paths implicit in the model (34) is not biased toward regions of convergence or divergence: $\int \nabla \cdot \mathbf{v} J \mathrm{d}^3\mathbf{a} = \int \nabla \cdot \mathbf{v} \mathrm{d}^3\mathbf{x} = 0$.

The stretching of an infinitesimal line element $\mathrm{d}x_i$ is unbounded, but that of a finite separation is bounded by the diameter of the domain D. The initially exponential growth of a finite separation must eventually saturate.

It remains to estimate the order of magnitude of the symmetrized rate-of-strain tensor e_{pq}, and the various time scales. Consider a Schrödinger wavefunction in the form of a sum of simple waves, propagating in one space dimension for simplicity. For example let $\psi = A \exp(-\mathrm{i}(\omega t + kx)) + B \exp(-\mathrm{i}(\omega t - kx))$, where A and B are unequal real constants, while ω and k are a real frequency and a real wave number, respectively. The gradient $\partial v/\partial x$ of the de Broglie velocity scales as $\hbar k^2/m \propto \omega$. The e-folding time of the path gradient $\partial P/\partial x$ therefore scales as $m/\hbar k^2 \propto 1/\omega$. The decorrelation time $T_\lambda = T_\lambda(\hbar, m, k)$ must also have the scale $m/\hbar k^2$. In general, the stretching rate is estimated to be $\Omega = E/\hbar$, where E is the expectation value of the energy for the state, and the stretching factor (38) is estimated to be $\exp(\Omega|t|)$. As demonstrated by Valentini and Westman [8], the fine grain in ρ appears within a few oscillations of the system.

3.3. Coarse graining

The fine grain in the evolving particle density $\rho[\mathbf{x}, t]$ may be suppressed by spatial smoothing or 'coarse graining':

$$\{\rho[\mathbf{x}, t]\} = l^{-3} \int_{\text{cell}} \rho[\mathbf{y}, t] \, \mathrm{d}^3\mathbf{y}, \tag{39}$$

where the region of integration is a small cubical cell centered on $\mathbf{x}$. The infinitesimal volume element $\mathrm{d}^3\mathbf{y}$ is $\mathrm{d}y^1 \, \mathrm{d}y^2 \, \mathrm{d}y^3$, and l is the length of the sides of the cubical cell. Such coarse graining is, again, always appropriate as the concept of a density defined precisely at a point is misleading. The density near a point is meaningful, being the mass per unit volume within a surrounding cell of vanishingly small but nonzero volume. In terms of the density ratio f, the coarse-grained particle density becomes

$$\{\rho[\mathbf{x}, t]\} = \{|\psi|^2[\mathbf{x}, t] f[\mathbf{x}, t]\} \approxeq |\psi|^2[\mathbf{x}, t]\{f[\mathbf{x}, t]\} \tag{40}$$

since, for very small cells, coarse graining has negligible impact on the very smooth Born density $|\psi|^2$. Again, coarse graining serves to suppress the chaotic behavior which is present in ρ but not present in $|\psi|^2$. The objective here is a proof that $\{f[\mathbf{x}, t]\} \sim 1$ as $t \to \infty$, or equivalently $\{g[\mathbf{x}, t]\} \sim 0$ as $t \to \infty$, where again g is the defect $f - 1$.

The first step toward a proof is the trivial result (27) that the QM expectation of the defect g vanishes identically at any time t:

$$\langle g\rangle \equiv \int_D g[\mathbf{x},t]|\psi|^2[\mathbf{x},t]\,\mathrm{d}^3\mathbf{x} = 0, \tag{41}$$

since both ρ and $|\psi|^2$ are normalized. This holds true in particular at time $t = 0$. Thus g takes values in D having both signs, equally with weight $|\psi|^2$. Consider next the coarse-grained defect. Substituting for $\mathrm{d}^3\mathbf{x}$ using (24) yields

$$\{g[\mathbf{x},t]\} \equiv l^{-3}\int_{\text{cell}} g[\mathbf{y},t]\,\mathrm{d}^3\mathbf{y} \approxeq l^{-3}|\psi|^{-2}[\mathbf{x},t]\int_{\mathcal{C}} g_0[\mathbf{b}]|\psi_0|^2[\mathbf{b}]\,\mathrm{d}^3\mathbf{b}, \tag{42}$$

since $|\psi|^2$ is smooth, that is, $|\psi|^2[\mathbf{y},t] \approxeq |\psi|^2[\mathbf{x},t]$ for all $\mathbf{y}$ in a cell with centroid $\mathbf{x}$. The region for the integration with respect to $\mathbf{b}$ is the image $\mathcal{C}$ of the cell under the mapping $\mathbf{b} = \mathbf{P}(\mathbf{y},t;0)$. If the cell image $\mathcal{C}$ were extended to the entire domain D, the second integral in (42) would be $\langle g_0\rangle$ which vanishes identically. Owing to line stretching as $t\to\infty$, the cell image does elongate indefinitely and tends to fill D. That is, $\{g\}\sim 0$ for large t, and so $\{\rho\}\sim|\psi|^2$: the coarse-grained particle density relaxes to QTE. A statistical estimate of the proximity to QTE is derived in the next section.

4. Relaxation variance

4.1. Sample statistics

It remains to establish the vanishing of $\langle\{g\}\rangle$, which is the QM expectation of the coarse-grained defect $\{g\}$, and also to estimate $\langle\{g\}^2\rangle - \langle\{g\}\rangle^2$ which is the QM variance of $\{g\}$. It is now necessary to specify a numerical implementation of the filter $\{\cdots\}$. Each coarse-graining cell is a cube of side $l \ll L$, fixed in space, with its centroid at a point $\mathbf{x}$ in a regular three-dimensional lattice spanning D. There are therefore $(L/l)^3$ cells, for integer values of $L/l \gg 1$. The cells are in turn partitioned into fixed cubic subcells of side l/N for some integer $N \gg 1$. The number N^3 of subcells in a cell must significantly exceed the number of chaotic disturbances in the cell, if the disturbances are to be adequately resolved and suppressed by the coarse graining. The subcells are identified by $\mathbf{n}$, an ordered triple of integers. The centroid of the $\mathbf{n}$th subcell is at $\mathbf{y_n}[\mathbf{x}] = \mathbf{x} + \mathbf{Y_n}$, where $\mathbf{Y_n}$ is independent of $\mathbf{x}$. Thus in each cell, the $\mathbf{n}$th subcell occupies the same position relative to the cell centroid. The cell average $\{\rho[\mathbf{x},t]\}$ is approximated by the arithmetic mean

$$\{\rho[\mathbf{x},t]\} = N^{-3}\sum_{\mathbf{n}} \rho[\mathbf{y_n},t], \tag{43}$$

with the expectation that the chaotic fluctuations of ρ within a cell cancel when summed with equal weight. The cell average (43) is no longer a moving average defined for every point $\mathbf{x}$, but is now only defined for centroids $\mathbf{x}$ on a lattice. The cell average should therefore be denoted as $\{\rho\}[\mathbf{x},t]$, etc, but the notation for the moving average will be retained. Valentini and Westman [8] use arithmetic means in partially overlapping cells, yielding smoother plots of $\{\rho[\mathbf{x},t]\}$.

A weighted sum over all cells in a coarse-grained field approximates the QM expectation for the fine-grained field. For example,

$$\langle\rho\rangle \approx l^3\sum_{\mathbf{x}}\{\rho[\mathbf{x}],t]\}|\psi|^2[\mathbf{x},t] = l^3N^{-3}\sum_{\mathbf{x}}\sum_{\mathbf{n}}\rho[\mathbf{y_n}[\mathbf{x}],t]|\psi|^2[\mathbf{x},t], \tag{44}$$

which may also be expressed as $\langle\{\rho[\mathbf{x}, t]\}\rangle$. Consider density defect which, being conserved on particle paths, satisfies

$$\{g[\mathbf{x}, t]\} = N^{-3} \sum_{\mathbf{n}} g[\mathbf{y_n}, t] = N^{-3} \sum_{\mathbf{n}} g_0[\mathbf{b_n}], \tag{45}$$

where $\mathbf{b_n} = \mathbf{P}(\mathbf{y_n}[\mathbf{x}], t; 0)$. That is, the uniformly weighted average (45) filters the chaotic fluctuations in the values of the initial defect sampled by the particle paths. By virtue of (27), and (44) applied to the fine-grained defect g, the QM expectation of the coarse-grained defect is found to vanish at time t and also initially:

$$\langle\{g\}\rangle = \langle\{g_0\}\rangle = 0. \tag{46}$$

The QM variance of the coarse-grained defect is

$$\langle\{g[\mathbf{x}, t]\}^2\rangle = l^3 \sum_{\mathbf{x}} \left(N^{-3} \sum_{\mathbf{n}} g[\mathbf{y_n}, t] \right)^2 |\psi|^2[\mathbf{x}, t], \tag{47}$$

which may be rearranged as

$$\langle\{g[\mathbf{x}, t]\}^2\rangle = l^3 N^{-6} \sum_{\mathbf{n},\mathbf{m}} \sum_{\mathbf{x}} g[\mathbf{y_n}[\mathbf{x}], t] g[\mathbf{y_m}[\mathbf{x}], t] |\psi|^2[\mathbf{x}, t], \tag{48}$$

which in turn maps into

$$\langle\{g[\mathbf{x}, t]\}^2\rangle = l^3 N^{-6} \sum_{\mathbf{n},\mathbf{m}} \sum_{\mathbf{x}} g_0[\mathbf{b_n}] g_0[\mathbf{b_m}] |\psi|^2[\mathbf{x}, t], \tag{49}$$

where again $\mathbf{b_n} = \mathbf{P}(\mathbf{y_n}[\mathbf{x}], t; 0)$. The sampled initial defect $g_0[\mathbf{b_n}]$ is now assumed to be 'white' in space, in the sense that

$$\sum_{\mathbf{x}} g_0[\mathbf{b_n}] g_0[\mathbf{b_m}] |\psi|^2[\mathbf{x}, t] = \delta_{\mathbf{n},\mathbf{m}} \sum_{\mathbf{x}} g_0^2[\mathbf{b_n}] |\psi|^2[\mathbf{x}, t]. \tag{50}$$

The 'white noise' assumption is justified, provided the separation $|\mathbf{y_n} - \mathbf{y_m}| = l/N$ of adjoining subcell centroids at time t stretches to $|\mathbf{b_n} - \mathbf{b_m}| > X$ at time 0, where X is the typical diameter of the subdomains in which g_0 is one-signed. The stretching factor is $\exp(\Omega|t|)$, where again $\Omega = |\mu| + \sigma_\lambda^2 T_\lambda \sim E/\hbar$, so $|t|$ must therefore exceed the relaxation time scale θ_N given by

$$\theta_N = \Omega^{-1} \ln(NX/l). \tag{51}$$

Combining (49) and (50) yields

$$\langle\{g[\mathbf{x}, t]\}^2\rangle = l^3 N^{-6} \sum_{\mathbf{x}} |\psi|^2[\mathbf{x}, t] \sum_{\mathbf{n}} g_0^2[\mathbf{b_n}], \tag{52}$$

that is,

$$\langle\{g[\mathbf{x}, t]\}^2\rangle = l^3 N^{-3} \sum_{\mathbf{x}} |\psi|^2[\mathbf{x}, t] \{g_0^2[\mathbf{P}(\mathbf{x}, t; 0)]\}. \tag{53}$$

Combining (28) and (53) with (24), where $\mathrm{d}^3\mathbf{x}$ is the cell volume l^3, yields the estimate

$$\langle\{g[\mathbf{x}, t]\}^2\rangle \sim N^{-3} \langle g_0^2 \rangle \tag{54}$$

for the QM variance of the coarse-grained defect, valid when $|t| > \theta_N$.

Once again, the preceding analysis establishes that at a sufficiently late time ($\Omega t \gg 1$), it is expected that the particles from different subcells of the same coarse-graining cell are initially scattered at random throughout the domain D, and so the initial defects g_0 for the particles have either sign at random. The initial particle density and hence also the initial defect are chosen freely of the de Broglie velocity and particle paths; thus, the initial values of defect sampled by the particles are unbiased. The defect is conserved on particle paths, and so the magnitude of the sample mean defect $\{g\}$ is greatly reduced by the many cancellations.

5. Remarks

5.1. Relaxation time scale

The stretching rate Ω is determined by Lagrangian moments of the principal Eulerian rates of strain of the velocity field. For example $\sigma_\lambda^2 T_\lambda$ is the integral of the autocovariance $\sigma_\lambda^2 C_\lambda(t)$. The latter is the Fourier transform of the frequency spectrum of the Lagrangian time series of a principal rate of strain. The derivation of the relaxation time scale, from the characteristics of the de Broglie velocity, is particularly pedantic. The principle is that more strain variance implies a greater stretching rate Ω and hence a shorter relaxation time θ_N. Also, the separation of the majority of subcell centroids at the coarse-graining time t is more like l than like l/N; hence, the relaxation time scale is more like $\theta_1 = \Omega^{-1}\ln(X/l)$.

5.2. H-functional

Valentini and Westman [8] define an H-functional by

$$H[g] = \langle (1+g)\ln(1+g)\rangle = \int_D \rho \ln(\rho/|\psi|^2)\,\mathrm{d}^3\mathbf{x}. \tag{55}$$

The value of $H[g]$ is a constant of the system: $H[g] = H[g_0]$. The value of $H[\{g\}]$ is not a constant, and $H[\{g\}] \approx \langle\{g\}^2\rangle$ for $\{g\} \approx 0$. Thus, $H[\{g\}] \sim N^{-3}\langle g_0^2\rangle$ as the percentage of the subcell centroids that are separated by X increases in time. The decay rate therefore is related to the cumulative log-normal distribution, but the inferred rate is only vaguely consistent with the approximately exponential decay of $H[\{g\}]$ found numerically by Valentini and Westman [8]. On the other hand, the separation cannot exceed the domain diameter and so the applicability of the log-normal distribution is restricted.

5.3. White initial defect

The assumption (50) of a white initial density defect $g_0 = \rho_0/|\psi_0|^2 - 1$ might seem unreasonable. The pair of particles at $\mathbf{b_n}$ and $\mathbf{b_m}$ at time 0 are both, at time t, in the cell with centroid $\mathbf{x}$. The particles move according to the field of de Broglie velocity $\mathbf{v}$, which is dynamically related to the Born density $|\psi|^2$ through the Born identity (4)–(6). The Born density defines the measure in (50). One the other hand, the position at time 0, of a particle labeled at time t, is the integral of the increasingly ($t \to \infty$) chaotic and effectively random velocities (10) along the particle path. The particle position at time 0 is therefore asymptotically normally distributed, but more importantly here it is asymptotically independent of the velocities along the path [24, 29]. Also, the case of interest here is that of an initial density ρ_0, and hence initial defect g_0, prescribed freely of the amplitude and phase of the wavefunction ψ. Particle separation assures that the sampled initial defects $g_0[\mathbf{b_n}]$ and $g_0[\mathbf{b_m}]$ can have different signs. It follows from the independence of the sampled initial defects that they are not correlated over the different cells (summation over $\mathbf{x}$).

Future computations following [8] and [21] should collect sample statistics, but the relaxation demonstrated in [8, 21] is strong evidence for the separation of particles, and also for the unbiasedness and whiteness of the initial defects sampled by the particles.

5.4. Choices of measure

The moments of separation are obtained from those of the symmetric part of the rate-of-strain tensor for the de Broglie velocity. The rate-of-strain tensor is not a Hermitian operator, being state dependent, so there is no QM precedent for the choice of measure. The measure $\mathrm{d}^3\mathbf{x}$ chosen here is uniform in space, in the Eulerian picture. In the Lagrangian picture, the measure

involves the Jacobian: $\mathrm{d}^3\mathbf{x} = J(\mathbf{a}, 0; t)\mathrm{d}^3\mathbf{a}$. The mean divergence vanishes ($\int \boldsymbol{\nabla} \cdot \mathbf{v}\mathrm{d}^3\mathbf{x} = 0$) and hence the analysis of separation is conclusive. While the moments of separation are based on the uniform measure derived from the phase of the wavefunction ψ, those of defect are based on the standard QM measure ($|\psi|^2[\mathbf{x}, t]\mathrm{d}^3\mathbf{x} = |\psi_0|^2[\mathbf{a}]\mathrm{d}^3\mathbf{a}$) derived from the wave amplitude. The defect is not Hermitian either, and another candidate measure would be the dBB measure ($\rho[\mathbf{x}, t]\mathrm{d}^3\mathbf{x} = \rho_0[\mathbf{a}]\mathrm{d}^3\mathbf{a}$). The mean defect vanishes and the variance of the fine-grained defect is conserved in the QM measure and also, if the defect were defined instead by $g = (f - 1)/f$, in the dBB measure. However the dBB measure is inappropriate since the particle density ρ is the 'hidden variable' under investigation here. On the other hand, the QM measure is of utility simply because the Born density constrains the de Broglie velocity through the Born identity (4). The Born density retains no conceptual role in de Broglie–Bohm mechanics.

5.5. Relative dispersion regimes

The moments of separation or relative dispersion at time 0 from a finite separation l/N at time t, or vice-versa, are assumed to be the same as those from an infinitesimal separation $\mathrm{d}x_i$. By analogy with isotropic and stationary turbulence, the moments are indeed the same if the squared magnitude of the spatial Fourier transform of the de Broglie velocity field, integrated over a spherical shell of a constant wave number magnitude k, is of the form $k^{-\alpha}$ where $\alpha > 3$ (see, e.g., [19]). It may assumed that this infrared catastrophe is cut off at $1/L$, where L is the domain width. In the case $\alpha > 3$, the separation at time r for $0 \leqslant r \leqslant |t|$ owes to 'eddies' in the de Broglie velocity field with a length scale k^{-1} much larger than the instantaneous separation $\Delta P(r) \equiv |\mathbf{P}(\mathbf{y_n}, t; r) - \mathbf{P}(\mathbf{y_m}, t; r)|$. If $\alpha < 1$, then separation owes instead to the smallest scales of motion, and the particles take independent random walks. Separation variance grows like $\mathcal{O}(|t|/T_v)$ as time runs forward or backward, where T_v is the decorrelation time for the velocity following a particle. This latter case ($\alpha < 1$) is very similar in essence to the diffusive model for relaxation proposed by Bohm and Hiley [1], and to the subquantum mechanics of Nelson [31]. There is a third case ($1 < \alpha < 3$) of great interest for a viscous fluid, in which the turbulent kinetic energy is transferred from low to high wave numbers at a constant rate ϵ. The transfer owes to the nonlinear advective acceleration, which has the Eulerian representation $\mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v}$ in configuration space. The kinetic energy, injected at low wave numbers at the rate ϵ by mechanical agitation, is dissipated at the same rate by viscous stresses at high wave numbers. In the intervening 'energy inertial range', the separation $\Delta P(r)$ is controlled by eddies of wave number $k \sim 1/\Delta P(r)$. The separation variance grows as $\epsilon|t|^3$ (see, e.g., [19]). The first observations of this range were made by Richardson [30], who tracked small balloons in the atmosphere. The 'self-similar' energy-cascading dynamics within the range are not relevant to the de Broglie velocity, unless the external and quantum potentials (see, e.g., [11]) can act as source and sink, respectively, in the wave number space representation of the quantum Hamilton–Jacobi equation.

In the two-dimensional domain considered by Valentini and Westman [8], the wavefunction ψ for the bounded mixed state includes only the first 16 modes; thus, the highest wave number is $2^{7/2}\pi/L$. This is not a description of the wave number spectrum of the de Broglie velocity $\mathbf{v} = (\hbar/m\mathrm{i})\boldsymbol{\nabla} \ln(\psi/|\psi|)$, but the assumption of exponentially growing relative dispersion owing to large-scale eddies is of the most interest here.

5.6. Stationarity and isotropy

The analysis of infinitesimal line stretching, as outlined in section 3, assumes that the stretching rate is unaffected by local diagonalization of the symmetrized rate-of-strain tensor at any point

on a particle path. Both the symmetrization and the diagonalization involve local rotations of coordinate axes. This implicitly requires statistical isotropy and stationarity of the Lagrangian velocity field.

The analysis of the stretching of finite line segments, as mentioned in the preceding subsection, involves an exact expression for the time rate of change of separation variance conditioned by instantaneous separation (the 'relative diffusivity') in terms of a Bessel transform of the nondirectional wave number spectrum of velocity. The validity of the exact expression depends upon the isotropy of the covariance tensor for the difference between the velocities at two points. Again, the expression may be simplified if the nondirectional wave number spectrum of velocity is steeper than k^{-3}. When the simplified relative diffusivity is combined with the Richardson–Kraichnan or master equation for separation, the distribution of separations is found to be the same log-normal distribution inferred in section 3. The derivation of the master equation also assumes that velocity differences decorrelate in time. The details may be found in [19]. The analysis of infinitesimal and finite separation presented here is not innovative and is well established in the theory of turbulence [32, 33].

Stationarity of the rate-of-strain statistics is assumed for simplicity. Subject to the classical Lindeberg condition, the central limit theorem holds also in the nonstationary case, that is, for sums of random variables from different populations [24, 29]. Provided that the moments of separation increase in time, and the initial defect is sampled without bias, the coarse-grained defect will vanish. That is, relaxation will occur. Similarly, isotropy is assumed for simplicity. Provided that there is a positive-definite relative diffusivity tensor, separation is statistically inevitable. Separation is eventually bounded by the domain, at which time local isotropy is certainly lost. By then, however, it is assured that the coarse-graining particles are randomly sampling different regions where the defects have different signs.

6. Loss of reversibility

Detailed information about initial positions of particles is destroyed by coarse graining at later times. The information is irrecoverable; thus the relaxation of coarse-grained particle densities is irreversible. The process of destruction may be analyzed as follows.

Consider a de Broglie velocity field $\mathbf{v}$ and a smooth initial particle density ρ_0. It will be convenient to refer to a flight time r, where $0 \leqslant r \leqslant t$. The density ρ_0 at time $r = 0$ is advected by $\mathbf{v}$, through the Eulerian prescription (12) or equivalently via the Lagrangian prescription (20), to the fine-grained particle density ρ at $r = t > 0$. Consider a subsequent reversal of time. Let ρ_R denote the 'reversed' density recovered at $r = 0$ from ρ at $r = t$. Particles are returned to their original positions; hence, $\rho_R = \rho_0$. Consider next the density $\{\rho\}_R$ recovered at $r = 0$ from $\{\rho\}$. The former is finely grained, even though the latter is the density at $r = t$ after coarse graining. Coarse graining again, now at $r = 0$, yields the smooth field $\{\{\rho\}_R\}$ which is close to the equilibrium $\{|\psi_0|^2\}$, but which is not necessarily close to $\{\rho_0\}$. The resolution of this paradox is that particles in a small cell for the second coarse graining (at $r = 0$) are widely separated at $r = t$, and particles in a small cell for the first coarse graining (at $r = t$) are widely separated at $r = 0$. Thus, $\{\{\rho\}_R\}$ is obtained from ρ_0 by two intervals of stirring. Again, as shown in section 3, the moments of separation grow exponentially as time advances backward or forward; hence, the reversed evolution from $r = t$ to $r = 0$ is statistically indistinguishable from a forward evolution from $r = t$ to $r = 2t$. Coarse graining and the statistical stretching of line elements regardless of the direction of time make quantum thermal equilibrium irreversible.

The statistical symmetry of stretching is most simply illustrated by considering the expectation of $\exp(at)$, where t is time in the interval $-\infty < t < \infty$ and the constant a

is a normal random variable with zero mean and unit variance. The expectation is readily shown to be $\exp(t^2/2)$. When the random exponent at is replaced with the time integral of a stationary random process, the expectation of the exponential is given by (38). Exponential growth is assured here for $|t| \to \infty$, as explained following (38). There is a straightforward extension of (38) to nonstationary processes.

Valentini and Westman [8] caution that if the particle density prescribed at time $r = 0$ were identical to the above-mentioned fine-grained density ρ at time $r = t$, the density would be advected by the de Broglie velocity field $-\mathbf{v}$ to the original smooth density ρ_0 at time t. The density ρ_0 may be far from QTE, even after coarse graining. Such a prescription of density at $r = 0$ is of course intricately related to subsequent values of the velocity. Yet, the exceptional can happen. Quoting from [8], 'Conceptually, the situation here is the same as in ordinary statistical mechanics'.

The coarse graining in the numerical computations reported in [8] is an arithmetical operation tuned to the computed chaos. The question arises: what is the nature of coarse graining in the real world? Valentini and Westman [8] argue that it is the 'violent' action of magnetic fields, etc, on the quantum system throughout the astrophysical lifetime of the system. That is, any real measurement will coarse grain the near-infinitesimal fine grain now present in the particle densities.

The de Broglie flow conserves the QM measure, as expressed in (24). This 'equivariance' [2] imposes on the particle motions the consequences of the Poincaré recurrence theorem [3, 21]. Each particle must eventually return arbitrarily closely to its initial position. The initial value of the defect g is conserved following the motion of the particle, and so is close to the initial value at the point of close return. By extension, all the particles in a coarse-graining cell must have been in that same cell at some time in the distant past. Stirring is ineffective in the long run, and QTE must eventually be lost. The conclusion is formally denied here since the spatial averaging substitutes for ensemble averaging, that is, particle separation is *probably* inevitable. In reality, no quantum system is isolated and thus the Poincaré recurrence time is that of the universe.

7. Counterexamples

There are several familiar quantum systems in which an arbitrary initial particle density does not relax to QTE. These systems will be described in outline.

7.1. Plane waves and wave packets

A plane wave solution of the Schrödinger equation (1) has a phase which is a linear function of position. The de Broglie velocity is therefore uniform in space, and so merely translates an initial particle density. More interesting is the well-known Gaussian wave packet solution (see, e.g., [11]). Given an initial wave number vector $\mathbf{k}$, the center of the packet has the velocity $\mathbf{u} = \hbar\mathbf{k}/m$, and for large time the packet width grows essentially linearly in time in all directions. Relative to an observer moving with the center of the packet, the de Broglie velocity is uniformly divergent in all directions and so merely expels particles radially outward from the packet. Neither of these systems would stir an arbitrary initial particle density to QTE. Yet these solutions are incomplete as descriptions of the history of isolated quantum systems, which are all created by physical processes involving other forms of matter. The wavefunction for the component of the system that emits the plane wave, or the wave packet, should also be considered. That other part of the wavefunction may separate from the plane wave, or the wave packet, and itself return to a stationary bound state. However, during the emission process, or

'adjustment' in the terminology of geophysical fluid dynamics, the combined wavefunction would be transient and so would have an intricate field of phase [2]. The adjusting field of de Broglie velocity would be far from simple and would be capable of stirring the particle density to QTE. The initial particle density should thus be taken as that at the onset of the establishment of the isolated packet, for example, and not some virtual perturbation from the QTE which is well established by the time the packet has become isolated.

As a proxy for the other matter in the creation of the isolated Gaussian packet, consider a second and identical particle. The configuration space is $\mathbb{R}^6$. Either particle may be in Alice's lab or in Bob's; hence, the spatial factor in the wavefunction is the symmetric sum of products of pairs of single-particle packets [11]. That is, the wavefunction is a pair of packets in $\mathbb{R}^6$ separating along a line in $\mathbb{R}^6$. It is readily shown that the interference region between the two centers expands linearly in time, for large time, and that the region extends off the line of separation. Thus a joint particle density between the packets is always being stirred.

The intuitively obvious distinction between a laminar or non-stirring field of de Broglie velocity, and a turbulent field that can stir, may be quantified as follows. The Gaussian wave packet for a free particle is an integral over the continuum of energy eigenstates, which would seem to imply that the de Broglie velocity is turbulent. However, the complex spectral coefficients are in precise relationship to one another, that is, the spectrum is coherent. The coefficients in the sums of eigenstates chosen by [8] and [21], on the other hand, are pseudo-random complex numbers. That is, their spectra are incoherent. Similarly, a regular train of cuspoidal water waves can have the same power spectrum as a random wave field, but the bispectra are different. In terms of the assumptions made here, the cuspoidal wave train is anisotropic and the Lagrangian time series of particle velocity do not decorrelate.

7.2. Stationary bound states

Colijn and Vrcsay [12] derive the de Broglie velocity for the electron of the hydrogen atom, in the singlet state with a constant spin factor. They find non-rigid axisymmetric flow around the direction of the spin eigenstate. This 'laminar' flow does not stir radially, nor azimuthally. On the other hand during the formation of the hydrogen atom, which quite likely took place in the early universe, the electron capture by the proton would entail the emission of a photon. The process is described not by quantum mechanics but by quantum electrodynamics (QED). It must be conceded that the extension of the dBB concepts to quantum field theory is not very far advanced, and there is certainly no clear picture of density stirring during a QED process. Scattering, nevertheless, serves as a proxy for capture. The spatial variations in the phases of scattering eigenfunctions (see, e.g., [34, 35]) are so intricate that the de Broglie velocities have an efficient stirring action.

7.3. Localized initial densities

If the initial particle density were highly localized, then the particles would take a long time to disperse throughout the domain of the mixed bound state considered in section 3. Quantum thermal equilibrium might not obtain until there have been many oscillations of the quantum system. Yet again the case is artificial. The particles are emitted from the source for the wavefunction that eventually adjusts to the bound state. Thus, there is in reality no disparity in the spatial extents of the particle density and the stirring field during adjustment.

The comments in this section are by no means proofs that $\{\rho\}$ relaxes to $|\psi|^2$ during adjustment, but they indicate the incompleteness of simple 'counterexamples'.

8. Discussion

The kinematics in sections 2 and 3 do not depend upon the dimensionality of $\mathbf{x}$, so relaxation also results from stirring by the de Broglie velocity associated with a Schrödinger wavefunction on the multi-particle configuration space $(\mathbf{x}_1, \mathbf{x}_2, \ldots)$. It is straightforward to extend the analysis to the single-particle Dirac equation for relativistic fermions with spin-1/2 [11], and to the single-particle Kemmer–Duffin–Petiau equation for relativistic bosons with spin-0 or spin-1 [36]. In each case, the quantum thermal equilibrium in the laboratory frame is $\psi^\dagger\psi\gamma$, where $\psi^\dagger\psi$ is defined and normalized in the rest frame of the particle, $\gamma = 1/\sqrt{1 - |\mathbf{v}|^2/c^2}$ is the Lorentz factor and $\mathbf{v}$ is the particle velocity. The factor allows for the Lorentz contraction of a volume element containing particles, in the direction of the motion of the particles [37]. A Lagrangian analysis of the de Broglie–Bohm mechanics of a quantum field would require that our understanding of relative dispersion in configuration space be extended to function space.

It has been argued [15] that early variations from QTE explain the cosmic horizon problem, and that the variations ought to be evident in the cosmic microwave background. The Planck Mission[3] is collecting new CMB data. Direct observation of relaxation to QTE in the laboratory would require resolving the relaxation time scale, which is about an atomic oscillation or about 1 fs. For a double-slit interference pattern to be non-Born, the times of flight would also have to be about 1 fs or less. The flight paths would have to be about 1/Rydberg.

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